ON STABILITY OF QUASIHARMONIC SYSTEMS WITH RETARDATION

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A method of calculation of characteristic indices is presented for systems of linear differential equations with time lag and with periodic coefficients close to constants. The criterion of stability of the unperturbed motion is established for these systems.

1. Formulation of the problem. We shall consider the stability of motion $x = 0$ of quasiharmonic systems of the type

$$\frac{dx(t)}{dt} = (a + \mu g(t, \mu))\, x(t) + (b + \mu h(t, \mu))\, x(t - \tau) \tag{1.1}$$

Here $x(t)$ is an n-dimensional vector; $a = \| a_{sk} \|$ and $b = \| b_{sk} \|$ are square matrices of the order n with constant elements; $g(t, \mu) = \| g_{sk}(t, \mu) \|$ and $h(t, \mu) = \| h_{sk}(t, \mu) \|$ are square matrices of the order n whose elements are analytic functions of the parameter μ in the region $|\mu| < \mu^*$ (μ^* being a positive number) and continuous functions of time t with the period 2π; τ is the constant retardation time.

The problem of stability in first approximation of periodic vibrations of quasilinear systems with retardation is reduced to a problem of this type.

Let us consider the characteristic equation

$$\Delta(\lambda) \equiv | a + be^{-\lambda\tau} - E\lambda | = 0 \tag{1.2}$$

where E is the unit matrix.

If $|\mu|$ is sufficiently small, the following propositions for the system (1.1) are valid, independently of the forms of the coefficients g and h. If all the roots of the characteristic equation (1.2) have negative real parts, then the motion $x = 0$ of the system (1.1) is asymptotically stable and any solution of (1.1) decreases exponentially [1,2]. If at least one root of Equation (1.2) has a positive real part, then $x = 0$ is unstable [3].

In case some of the roots of Equation (1.2) have real parts equal to zero (the remaining roots having negative real parts), the stability of motion is essentially influenced by the functions $g(t, \mu)$ and $h(t, \mu)$.

This paper deals with a method of solution of the problem of stability of the motion $x = 0$ of the system (1.1) in the indicated case.

It is shown that, as in the case of ordinary differential equations with periodic coefficients, for every root λ_1 of Equation (1.2) a particular solution can always be constructed in the form

$$x_s(t) = e^{\alpha t} u_s(t) \qquad (\alpha(0) = \lambda_1) \tag{1.3}$$

where $\alpha(\mu)$ is a constant with respect to t, and $u_s(t, \mu)$ are periodic functions with the period 2π. The constant $\alpha(\mu)$ will be called the characteristic index.

Unlike the ordinary differential equations, the system (1.1) has a countable number of characteristic exponents: as many as the number of roots of the characteristic equation (1.2). However, as in the case of ordinary differential equations with periodic coefficients close to constant, it is not necessary to determine all the characteristic indices in order to clarify the question of stability or instability of the motion $x = 0$ of the system (1.1). It is sufficient to determine only those characteristic indices which correspond to the roots of Equation (1.2) with zero real parts. Assume that all the characteristic indices corresponding to pure imaginary roots of Equation (1.2) are determined and suppose their real parts are negative. Then the motion $x = 0$ of the system (1.1) is asymptotically stable for sufficiently small $|\mu|$, and all solutions of the system (1.1) will decrease exponentially. If at least one of the characteristic indices will have a positive real part, then the motion $x = 0$ is unstable, because there exists one particular solution of the system (1.1) in the form (1.3) which increases exponentially.

This paper is concerned with the proof of this last proposition and with the problem of calculation of characteristic indices.

In the following, a method of calculation of characteristic indices and particular solution of the type (1.3) will be presented for the system (1.1). This method is a generalization for the system (1.1) of a known method of calculation of characteristic indices of systems of ordinary linear differential equations with periodic coefficients close to constants [5,6,7].

2. The method of calculation of characteristic indices.

Let us consider an arbitrary root λ_1 of Equation (1.2). The root λ_1 is simple with respect to $i = \sqrt{-1}$ if all the differences $\lambda_1 - \lambda_j (j = 2, 3, \ldots)$ are not equal to zero or a number of the type $\pm Ni$, N being a positive integer. This property will be briefly denoted by: $\lambda_1 \not\equiv \lambda_j (\mathrm{mod}\ i)$ $(j = 2, 3, \ldots)$.

The roots $\lambda_1, \ldots, \lambda_m$ are equal to each other in the modulus i and are different in the modulus i from the remaining roots $\lambda_{m+1}, \ldots$ of the characteristic equation (1.2) if $\lambda_1 \equiv \lambda_2 \equiv \ldots \equiv \lambda_m$, $\lambda_k \not\equiv \lambda_j (\mathrm{mod}\ i)$ for $k \leqslant m$, $j \geqslant m + 1$, where k and j are integers.

In the calculation of the characteristic index $\alpha_1(\mu)$ corresponding to the root λ_1, two cases may exist, depending on whether the root λ_1 is simple with respect to the modulus i or is a multiple of certain multiplicity m (m is finite for systems with retardation).

Assume that λ_1 is a simple root in the modulus i of the characteristic equation (1.2). The corresponding characteristic index $\alpha_1(\mu)$ will be an analytical function of the parameter μ in a certain, sufficiently small, surrounding of the point $\mu = 0$. In order to calculate the characteristic index $\alpha_1(\mu)$, we shall seek the particular solution in the form (1.3), where

$$\alpha_1(\mu) = \lambda_1 + \mu\alpha_1 + \mu^2\alpha_2 + \ldots, \qquad v_s(t, \mu) = u_s^0(t) + \mu u_s^{(1)}(t) + \ldots \tag{2.1}$$

$\alpha_1, \alpha_2, \ldots$ are unknown constants, and $u_s^{(0)}, u_s^{(1)}, \ldots$ are unknown non-periodic coefficients. Substituting (1.3) into Equations (1.1) and taking into account (2.1), we obtain the equations

$$\frac{du^{(0)}}{dt} = au^{(0)}(t) + bu^{(0)}(t - \tau)\, e^{-\lambda_1\tau} - \lambda_1 u^{(0)}(t) \tag{2.2}$$

$$\frac{du^{(1)}(t)}{dt} = au^{(1)}(t) + bu^{(1)}(t - \tau)\, e^{-\lambda_1\tau} - \lambda_1 u^{(1)}(t) + g(t, 0)\, u^{(0)}(t) + \\ + h(t, 0)\, e^{-\lambda_1\tau} u^{(0)}(t - \tau) - \alpha_1 u^{(0)}(t) - bu^{(0)}(t - \tau)\, e^{-\lambda_1\tau}\tau\alpha_1 \tag{2.3}$$

Let us investigate the system of equations (2.2). Since the characteristic equation of this system has one root equal to zero and the remaining roots $\lambda_j - \lambda_1 (j = 2, 3, \ldots)$ are different from zero and numbers of the type $\pm Ni$, $N = 1, 2, 3, \ldots$, the only periodic solution with the period 2π is the constant vector $x = \phi_1 = \{\phi_{s1}\} = \mathrm{const}$. We denote by $\Delta_{kj}(\lambda_1)$ the algebraic complement of the element of the kth row and jth column of the determinant $\Delta(\lambda_1)$, (1.2). Without limiting the generality, we shall consider that $\Delta_{11}(\lambda_1)$ is not equal to zero. Thus we obtain

$$\varphi_1 = \{c\Delta_{11}(\lambda_1), \ldots, c\Delta_{1n}(\lambda_1)\} \tag{2.4}$$

where c is an arbitrary constant.

Consider now the system conjugate to (2.2)

$$\frac{dy(t)}{dt} = -a'y(t) - b'e^{-\lambda_1\tau}y(t+\tau) + \lambda_1 y(t) \tag{2.5}$$

where a' and b' are the transposes of the matrices a and b. The characteristic equation of the system (2.5)

$$D(\lambda) \equiv |-a' - b'e^{-\lambda_1\tau+\lambda\tau} - E(\lambda - \lambda_1)| = 0 \tag{2.6}$$

has one root equal to zero while the remaining roots $-\lambda_j + \lambda_1 (j = 2, 3, \ldots)$ are different from zero. The single periodic solution of the system (2.5) is

$$\psi_1 = \{\Delta_{11}(\lambda_1), \ldots, \Delta_{n1}(\lambda_1)\} \quad \Delta_{11}(\lambda_1) \neq 0 \tag{2.7}$$

We shall substitute $u^{(0)} = \phi_1$ into the system (2.3), and we shall seek its periodic solution $u^{(1)}(t)$ with the period 2π. In order that this solution may exist, the following condition should be satisfied:

$$-\alpha_1 \int_0^{2\pi} (\varphi_1 c + bc\varphi_1(t-\tau)e^{-\lambda_1\tau}\tau)\varphi_1\,dt + \\ + \int_0^{2\pi} [g(t,0)c\varphi_1 + h(t,0)c\varphi_1 e^{-\lambda_1\tau}]\varphi_1(t)\,dt = 0 \tag{2.8}$$

It is easy to show that

$$\int_0^{2\pi} (\varphi_1 + b\varphi_1 e^{-\lambda_1\tau}\tau)\psi_1\,dt = -2\pi c\Delta_{11}(\lambda_1)\frac{d\Delta(\lambda)}{d\lambda}\Big|_{\lambda=\lambda_1} \neq 0 \tag{2.9}$$

Therefore, the coefficient of a_1 in Equation (2.8) is not equal to zero. Considering (2.9), we find

$$\alpha_1 = -\frac{1}{2\pi}\frac{1}{c\Delta_{11}(\lambda_1)\Delta'(\lambda_1)}\int_0^{2\pi} [g(t,0)\varphi_1 + h(t,0)\varphi_1]\psi_1 dt \tag{2.10}$$

Using the method of undetermined coefficients, we find the periodic solution $u^{(1)}(t)$ of the system (2.3) in the form

$$u^{(1)}(t) = c_1\varphi_1 + U^{(1)}(t)\,c \tag{2.11}$$

The periodic vector will be completely determined if we require that the condition $v_1^{(1)}(0) = 0$ be satisfied. All the following unknown coefficients $a_2, a_3, \ldots, u^{(2)}, u^{(3)}, \ldots$ will be found in the same order as a_1, $u^{(1)}$. All $u^{(k)}$ will be fully determined if the condition $u_1^{(k)}(0) = 0$ $(\Delta_{11}(0) \neq 0)$ is added.

With this additional condition imposed on the initial values of the functions $u_1^{(k)}(0)$, the series $u_s(t, \mu)$ will converge for sufficiently small $|\mu|$, while the series for the characteristic index always converge for sufficiently small $|\mu|$.

We shall consider now the case of a root λ_1 of Equation (1.2) which is multiple in the modulus i. We shall again seek the solution of the system (1.1) in the form (1.3), where $\alpha_1(\mu)$ and $u(t, \mu)$ are represented as the series (2.1) with unknown coefficients. Substituting (1.3) into the system (1.1) and comparing the coefficients of equal powers of μ, we obtain the equations (2.2), (2.3), ...

Consider the system of equations (2.2). The characteristic equation of this system has m roots equal to zero in the modulus i. Assume that m periodic solutions ϕ_j $(j = 1, \ldots, m)$ of the period 2π correspond to these roots. Thus, also the conjugate system (2.5) admits m periodic solutions ϕ_j $(j = 1, \ldots, m)$ with the period 2π.

We assume

$$U^{(0)} = M_1^{(0)}\varphi_1(t) + \ldots + M_m^{(0)}\varphi_m(t) \tag{2.12}$$

where $M_1^{(0)}, \ldots, M_m^{(0)}$ are arbitrary constants.

Substituting $u^{(0)}$ into Equations (2.3), we obtain the equations for the determination of the periodic vector function $v^{(1)}$ with the period 2π. In order that this system of equations may admit periodic solutions of the period 2π, the following conditions should be satisfied:

$$-\alpha_1 \int_0^{2\pi} [M_1\varphi_1 + \ldots + M_m\varphi_m + b\,(M_1\varphi_1(t-\tau) + \ldots$$
$$\ldots + M_m\varphi_m(t-\tau)\,e^{-\lambda_1\tau}]\,\psi_j dt +$$
$$+ \int_0^{2\pi} [g(t, 0)\,(M_1^{(0)}\varphi_1(t) + \ldots + M_m^{(0)}\varphi_m(t)) +$$
$$-\,h(t, 0)\,(M_1^{(0)}\varphi_1(t-\tau) + \ldots + M_m^{(0)}\varphi_m(t-\tau))\,e^{-\lambda_1\tau}]\,\varphi_j(t)\,dt$$

or

$$-\alpha_1(d_{j1}M_1^{(0)} + \ldots + d_{jm}M_m^{(0)}) + Q_{1j}M_1^{(0)} + \ldots + Q_{mj}M_m^{(0)} = 0 \tag{2.13}$$

The necessary and sufficient condition for the system (2.13) to admit nontrivial solution is that its determinant equal to zero

$$|-\alpha_1 d_{jk} + Q_{kj}| = 0 \tag{2.14}$$

We shall show first that (2.14) is an equation of the order m in α_1, since $|d_{jk}| \neq 0$. Let us assume that $|d_{jk}| = 0$. Then, such constants Λ_1, $\ldots$, Λ_m may be found that

$$\Lambda_1 d_{j1} + \ldots + \Lambda_m d_{jm} = 0 \qquad (j = 1 \ldots, m)$$

But then the system (2.1) admits solutions with the secular term $(\Lambda_1\phi_1 + \ldots + \Lambda_m\phi_m)t + \phi(t)$. This last case is excluded by the assumption, because to m roots being equal in the modulus i correspond m independent periodic solutions (2.1); in particular, if the roots $\lambda_1, \ldots, \lambda_m$ are equal in the modulus i, but all are different, then

$$d_{jj} = -2\pi\Delta_{11}(\lambda_j)\frac{d\Delta(\lambda)}{d\lambda}\bigg|_{\lambda=\lambda_j}, \qquad \Delta_{11}(\lambda_j) \neq 0$$

Thus, Equation (2.14) is of the order m and admits m roots $a_1^{(1)}, \ldots, a_1^{(m)}$. In the case when $a_1^{(k)}$ are all different, the m characteristic indices correspond to them:

$$\lambda_j + \mu\alpha_1^{(j)} + \mu^2(\ldots) + \ldots \qquad (2.15)$$

In the case when multiple roots are among the roots $a_1^{(1)}, \ldots, a_1^{(m)}$, the quantities $\lambda_j + a_1^{(j)}\mu$ $(j = 1, \ldots, m)$ represent first approximations of the characteristic indices corresponding to m roots $\lambda_1, \ldots, \lambda_m$ being equal in the modulus i.

We shall discuss in more detail the calculation of the characteristic indices corresponding to the simple roots a_1 of Equation (2.14). We denote by $D_{kj}(a_1)$ the algebraic complement of the element of the kth row and the jth column of the determinant (2.14). Since

$$\frac{dD(\alpha)}{d\alpha}\bigg|_{\alpha=\alpha_1} \neq 0, \qquad \text{for } \alpha = \alpha_1$$

then, without impairing the generality, we may assume that

$$D_{mm}(\alpha_1) \neq 0$$

Solving the system (2.12) with respect to $M_j^{(0)}$, we find

$$M_j^{(0)} = \frac{D_{mj}(\alpha_1)}{D_{mm}(\alpha_1)}c \qquad (j = 1, \ldots, n)$$

where c is an arbitrary constant.

Since the conditions of existence of periodic solutions of the system (2.3) are thus satisfied, a periodic solution with the period 2π exists and it can be determined using the method of undetermined coefficients

$$u^{(1)} = M_1^{(1)}\varphi_1(t) + \ldots + M_m^{(1)}\varphi_m(t) + c\Phi^{(1)}(t) \qquad (2.16)$$

where $\Phi(t) = \{\Phi_s(t)\}$ are periodic functions, and $M_1^{(j)}$ are arbitrary constants. Substituting $u^{(1)}(t)$ into the equations for $u^{(2)}$, we obtain the conditions of existence of periodic solution in the form

$$-\alpha_2(d_{j1}M_1^{(0)}+\ldots+d_{jm}M_m^{(0)})+Q_{1j}M_1^{(1)}+\ldots+Q_{mj}M_m^{(1)}- \tag{2.17}$$
$$-\alpha_1(d_{j1}M_1^{(1)}+\ldots+d_{jm}M_m^{(1)})+cA_j^{(1)}=0 \qquad (j=1,\ldots,m)$$

Here $A_j^{(1)}$ are certain constants, Q_{ij} are defined in (2.13).

The system (2.17) represents m equations with respect to $m+1$ unknowns. We write the determinant of the system (2.17) with respect to the unknowns $M_1, M_2, \ldots, M_{m-1}, \alpha_2$

$$\begin{vmatrix} Q_{11}-d_{11}\alpha & \ldots & Q_{1m-1}-d_{1m-1}\alpha, & d_{11}M_1^{(0)}+\ldots+d_{1m}M_1^{(0)} \\ \ldots & \ldots & \ldots & \ldots \\ Q_{m1}-d_{m1}\alpha & \ldots & Q_{mm-1}-d_{mm-1}\alpha, & d_{m1}M_1^{(0)}+\ldots+d_{mm}M_m^{(0)} \end{vmatrix} = cD'(\alpha_1)\neq 0$$

The system (2.17) can be solved with respect to $M_1^{(1)}, \ldots, M_{m-1}^{(1)}$, α_2 which will be thus expressed in terms of the parameter c and the constant $M_m^{(1)}$. In this, α_2 proves to be independent from $M_m^{(1)}$ and c, and it is given by the formula

$$\alpha_2=\frac{1}{D'(\alpha_1)}\begin{vmatrix} Q_{11}-\alpha_1 d_{11} & \ldots & Q_{1m-1}-\alpha_1 d_{1m-1} & A_m^{(1)} \\ \ldots & \ldots & \ldots & \ldots \\ Q_{m1}-\alpha_1 d_{m1} & \ldots & Q_{mm-1}-\alpha_1 d_{mm-1} & A_m^{(1)} \end{vmatrix} \tag{2.18}$$

We find $u^{(2)}$, $u^{(3)}$, ... in a similar way as we did $u^{(1)}$. In this, the constants $M_1^{(k)}, \ldots, M_{m-1}^{(k)}$, α_{k+1} satisfy a system which differs only in free terms from the system (2.17). Later on, in Section 3, the analyticity of the characteristic index $\alpha_1(\mu)$ will be shown in the vicinity of the point $\mu = 0$ for any simple root α_1 of Equation (2.14). Therefore, from the uniqueness of the series presentation $\alpha(\mu) = \lambda_1 + \mu\alpha_1 + \ldots$, we conclude that the series for the characteristic index converges for sufficiently small $|\mu|$ and it represents the characteristic index being sought.

Without limiting the generality of discussion, we may assume that, for instance,

$$\sum_{j=1}^{m} M_j^{(0)}\varphi_{ej}(0)\neq 0 \quad \text{for } s=l$$

Thus, the selection of the constants $M_m^{(k)}$ will not be arbitrary if we require that the condition $u_e(0, \mu) = 1$ be satisfied at the initial instant of time.

This will be true if the following conditions are fulfilled:

$$u_e^{(0)}(0)=1, \qquad u_e^{(j)}(0)=0 \qquad (j=1,2,3,\ldots)$$

The first condition is fulfilled by assuming

$$c = \left(\sum_{j=1}^{m} \frac{D_{mj}(\alpha_1)}{D_{mm}(\alpha_1)} \varphi_{ej}{}^{(0)} \right)^{-1}$$

The other conditions will be satisfied if the constants $M_m^{(k)}$ are chosen in a proper way with regard to the condition $u_e^{(j)}(0) = 0$.

With such constants c, $M_m^{(1)}$, $M_m^{(2)}$, ... the series for the particular solution (1.3) corresponding to the roots λ_1 and α_1 will converge for sufficiently small $|\mu|$.

3. On the construction of particular solutions of the type (1.3) in a general case. We shall find particular solutions of the system (1.1) in the form (1.3). For this purpose, we replace the variable vector x in the system (1.1) by the vector u, according to the relation

$$x = e^{(\lambda_1+\alpha)}u \tag{3.1}$$

With the new variables the system (1.1) has the form

$$\frac{du(t)}{dt} = au(t) + bu(t-\tau)e^{-\lambda_1\tau} - \lambda_1 u(t) + \mu g(t,\mu)u(t) +$$
$$+ \mu h(t,\mu)u(t-\tau)e^{-\lambda_1\tau} + e^{-\lambda_1\tau}bu(t-\tau)(e^{-\alpha\tau}-1) +$$
$$+ \mu h(t,\mu)u(t-\tau)e^{-\lambda_1\tau}(e^{-\alpha\tau}-1) - \alpha u(t) \tag{3.2}$$

where a is an undetermined constant.

Instead of looking for particular solutions (1.3) of the system (1.1), we shall find periodic solutions with the period 2π of the system (3.2). Then a will be determined from the condition of existence of periodic solution of the system (3.2), which for $\mu = 0$ and $a = 0$ becomes the periodic solution (2.12) of the system (2.2).

Let us consider the system of integro-differential equations

$$\frac{du(t)}{dt} = au(t)+bu(t-\tau)e^{-\lambda_1\tau} - \lambda_1 u(t) + g(t,\mu)\mu u(t) + \tag{3.3}$$
$$+ \mu h(t,\mu)u(t-\tau)e^{-\lambda_1\tau} + be^{-\lambda_1\tau}u(t-\tau)(e^{-\alpha\tau}-1) +$$
$$+ \mu h(t,\mu)u(t-\tau)e^{-\lambda_1\tau}(e^{-\alpha\tau}-1) + \sum_{j=1}^{m}(\varphi_j(t) + be^{-\lambda_j\tau}\tau\varphi_j(t-\tau))W_j$$

$$\int_0^{2\pi}[\mu g(t,\mu)u(t) + \mu h(t,\mu)u(t-\tau)e^{-\lambda_1\tau} + bu(t-\tau)e^{-\lambda_1\tau}(e^{-\alpha\tau}-1) -$$
$$- \alpha u(t) + \mu h(t,\mu)u(t-\tau)e^{-\lambda_1\tau}(e^{-\alpha\tau}-1)]\,\psi_j(t)\,dt + \sum_{k=1}^{m} d_{jk}W_k = 0 \tag{3.4}$$

The discussion of the last system is based on the same assumption concerning the root λ_1 as in the second case of Section 2. The notations $\psi_j(t)$ and d_{jk} are explained in Section 2. The constants W_j are determined by Equations (3.4). The following lemma is true.

Lemma 1. The system of integro-differential equations (3.3) and (3.4) admits periodic solution with the period 2π, depending on the arbitrary constants $M_1, \ldots, M_m$, and the parameters μ and a, in the vicinity of the point $\mu = a = 0$, which for $\mu = a = 0$ becomes the periodic solution (2.12) of the system (2.2). This solution is linear and homogeneous with respect to the constants $M_1, \ldots, M_m$, the coefficients being analytical with respect to μ and a in the vicinity of $\mu = a = 0$.

Proof. We shall use the method of successive approximations. Let us assume that the periodic solution of the auxilliary system has been found. It is of the form

$$x^* (t, M, \ldots, M_m, \mu, a) = v_1 (t, \mu, a) M_1 + \ldots + v_m(t, \mu, a) M_m \quad (3.5)$$

where $v_1, \ldots, v_m$ are analytic in μ and a. The constants W_j^* are then uniquely determined by Equations (3.4).

Consider the system of equations

$$W_j^* (M_1, \ldots, M_m, \mu, a) \equiv P_{j1}(\mu, a) M_1 + \ldots + P_{jm} (\mu, a) M_m = 0 \quad (j = 1, \ldots, m) \quad (3.6)$$

Lemma 2. In order that the system (3.2) may admit a periodic solution of the period 2π, it is necessary and sufficient that the system of equations (3.6) have nontrivial solution for $M_1, \ldots, M_m$.

Proof. If $M_1^*, \ldots, M_m^*$ satisfy the system of equations (3.6), then substituting them into (3.5) we obtain the periodic solution of the system (3.2). Hence follows the sufficiency of the conditions of the lemma. To prove the necessity of the conditions of the lemma, we note that all the periodic solutions with the period 2π of the auxilliary system are included in Formula (3.5). We assume that the system (3.2) has a periodic solution $\phi(t, \mu, a)$ with the period 2π at certain fixed values of a and μ. Since this solution is also a periodic solution of the auxilliary system of integro-differential equations (3.3) and (3.4), the constants $M_1, \ldots, M_m \mid a, \mu$ may be found such that, if they are substituted into (3.5), the identities

$$\varphi (t, \mu, \alpha) \equiv v_1 (t, \mu,\alpha) M_1 (\mu, \alpha) + \ldots + v_m (t, \mu, \alpha) M_m (\mu, \alpha)$$

exist and, therefore, $M_1(\mu, a), \ldots, M_m(\mu, a)$ satisfy the system of equations (3.6).

The system of equations (3.6) admits a nontrivial solution if its determinant is equal to zero:

$$|P_{ij}(\mu, \alpha)| = 0 \tag{3.7}$$

This last equation has, in the vicinity of the point $\mu = 0$, m solutions $\alpha_1(\mu)$, ..., $\alpha_m(\mu)$ which satisfy the conditions $\alpha_j(0) = 0$ ($j = 1$, ..., m).

Adding the root λ_1 to $\alpha_j(\mu)$ ($j = 1$, ..., m), we obtain m characteristic indices corresponding to the roots λ_1, ..., λ_m equal to each other in the modulus i. We note that the characteristic indices of the system (1.1) are determined up to a constant term $\pm Ni$ (N being an integer or zero).

Performing the necessary calculations, we find that

$$|P_{kj}(\mu, \alpha\mu)| \equiv \mu^m |Q_{kj} - d_{kj}a| + \mu^{m+1}\Phi^*(\mu, \mu a) = 0 \qquad (\alpha = \mu a) \tag{3.8}$$

Here $\Phi^*(\mu, a)$ is an analytic function of the arguments μ and a in the vicinity of the point $\mu = 0$. Hence, on the basis of the theorem of implicit functions, we conclude that to each simple root of Equation (2.14) there corresponds a characteristic index, analytic in the vicinity of the point $\mu = 0$. It follows also from the form of (3.8) that the quantities $\lambda_j + \mu a_j^{(1)}$ ($j = 1$, ..., m) represent approximate expressions for the sought characteristic index. With Equation (3.7) established, the problem of determination of characteristic indices corresponding to the root λ_1 reduces to the solution of Equation (3.7) in the vicinity of the point $\mu = \alpha = 0$.

4. On the stability of motion $x = 0$ of the system (1.1).

We assume that m roots of the characteristic equation (1.2) have real parts equal to zero, while the remaining roots have negative real parts. We shall calculate the corresponding characteristic indices in the described manner. If at least one characteristic index has its real part positive, the motion is unstable, because the system (1.1) has a particular solution of the type (1.3) which increases exponentially. If all m characteristic indices have negative real parts, the motion $x = 0$ of the system (1.1) will be asymptotically stable, because any solution of the system (1.1) will decrease like an exponential function whose exponent is equal to the largest real part of the characteristic indices. The validity of this last proposition will be shown in this section.

Let $x(t)$ be the solution of the system (1.1). As an element of the solution we shall consider segments of the trajectories of the system (1.1) in the interval $[t, t - r]$. In the functional space of continuous functions the motion of the system (1.1) is determined by the vector functions of time $x_t(\vartheta) = x(t+\vartheta)$, $-r \leqslant \vartheta \leqslant 0$. In the functional space $x(\vartheta)$, to the linear system (1.1) there corresponds the linear system of "ordinary" differential equations with the operator right-hand side

$$\frac{dx_t(\vartheta)}{dt} = Ax_t(\vartheta) + \mu R(x_t(\vartheta)) \tag{4.1}$$

where

$$Ax(\vartheta) = \begin{cases} \dfrac{dx(\vartheta)}{d\theta} & (-\tau \leqslant \vartheta < 0) \\ ax(0) + bx(-\tau) & (\vartheta = 0) \end{cases} \tag{4.2}$$

$$R(x, (\vartheta)) = \begin{cases} 0 & (-\tau \leqslant \vartheta < 0) \\ g(t, \mu)\, x(0) + h(t, \mu)\, x(-\tau) & (\vartheta = 0) \end{cases} \tag{4.3}$$

We assume that to each of m roots of Equation (1.2) with zero real part there corresponds a periodic solution $\phi_j(t)$ with the period $2\pi/\omega_j$, where $\omega_j i = \lambda_j$ $(j = 1, \ldots, m)$.

Let us consider the conjugate system

$$\frac{dy_t(\vartheta)}{dt} = A^* y_t(\vartheta) \tag{4.4}$$

where

$$A^* y(\vartheta) = \begin{cases} \dfrac{dy(\vartheta)}{d\vartheta} & (0 < \vartheta \leqslant \tau) \\ -a^* y(0) - b^* y(\tau) & (\vartheta = 0) \end{cases} \tag{4.5}$$

a^* and b^* being the transposed matrices of a and b. The system (4.4) has also m periodic solutions $\psi_j(t)$ $(j = 1, \ldots, m)$.

For arbitrary two solutions $x_t(\vartheta)$ of the system (4.1), with $y_t(\vartheta)$ of the system (4.4), the following condition holds:

$$(x_t(\vartheta)\, y_t(\vartheta)) \equiv \sum_{j=1}^{n} x_{jt}(0)\, y_{jt}(0) - \sum_{j=1,\, l=1}^{n} \times \int_0^{-\tau} x_{et}(\xi)\, y_{jt}(\tau + \xi)\, b_{je}\, d\xi = \text{const} \tag{4.6}$$

Let $d_{ij} = (\phi_i(t+\vartheta)\psi_j(t+\vartheta))$. With the assumptions made for the roots λ_j $(j = 1, \ldots, m)$ of the characteristic equation (1.2), it is $|d_{ij}| \neq 0$.

We divide the space $x(\vartheta)$ into m-dimensional space $l : \{y_j\}$, and a functional subspace L with the index m, by assuming

$$\begin{gathered} x(\vartheta) = z(\vartheta) + \varphi_1(t + \vartheta_1)\, y + \ldots + \varphi_m(t + \vartheta)\, y_m \\ (z(\vartheta)\, \psi_j(t + \vartheta)) = 0 \qquad (j = 1, \ldots, m) \end{gathered} \tag{4.7}$$

In this, the variables $y_1, \ldots, y_m$ are uniquely determined from the system of equations

$$(x(\vartheta)\,\psi_j(t+\vartheta)) = d_{j1}y_1 + \ldots + d_{jm}\,y_m \qquad (j=1,\ldots,m) \tag{4.8}$$

Since L and L do not contain common elements, with the exception of the point $x(\vartheta) = 0$, (4.7) and (4.8) assure the uniqueness of the presentation (4.7).

In the variables $y_1, \ldots, y_m$, and $z(\vartheta)$, the system of equations (4.1) has the form

$$\frac{dy_k}{dt} = \mu \sum_{j=1}^{m} \frac{\Delta_{kj}}{\Delta}\,(R\,(x_j(\vartheta)), \psi_i(t+\vartheta)) \qquad (k=1,\ldots,m) \tag{4.9}$$

$$\frac{dz_t(\vartheta)}{dt} = Az_t(\vartheta) + \mu R\,(x_t(\vartheta)) - \mu \sum_{k=1}^{m} \varphi_k(t+\vartheta) \sum_{j=1}^{m} \frac{\Delta_{kj}}{\Delta}\,(R, \psi_j(t+\vartheta)) \tag{4.10}$$

where $\Delta = |d_{kj}|$; Δ_{kj} is the algebraic complement of the element d_{kj} of the kth row and jth column; on the right-hand sides of (4.9) and (4.10), $x_t(\vartheta)$ should be assumed as given by Formulas (4.7).

We assume now that to the roots $\lambda_1, \ldots, \lambda_m$ of Equation (1.2) correspond different characteristic indices $\alpha_j(\mu)$ and particular solutions of the system (1.1) in the form (1.3) ($j = 1, \ldots, m$). Thus, the system (4.9) and (4.10) admits m independent particular solutions

$$y_j^{(e)}(t) = e^{\alpha_e(\mu)\,t}\,v_j^{(e)}(t,\mu), \qquad Z_t^{(e)}(\vartheta) = e^{\alpha_e(\mu)\,(t+\vartheta)}\,w_t^{(e)}(\vartheta,\mu)$$
$$(e=1,\ldots,m)\ (j=1,\ldots,m) \tag{4.11}$$

Let us consider an arbitrary particular solution of the system (1.1) or (4.1), to which corresponds the particular solution $\{y_j(t),\ z_t(\vartheta)\}$ of the system (4.9) and (4.10). The functions $y_j(t)$ can be represented as linear combinations of m functions $y_j^{(e)}(t)$ entering into m particular solutions (4.11)

$$y_j(t) = c_1 y_j^{(1)}(t) + \ldots + c_m\,y_j^{(m)}(t) \tag{4.12}$$

where $c_1, \ldots, c_m$ are certain constants. Therefore, the functions $y_j(t)$ in an arbitrary particular solution of the system (4.9), (4.10) decrease as the exponential function with the exponent equal to the real part of one of the characteristic indices $\alpha_j(\mu)$. Substituting the $y(t)$ found into the system (4.10), we obtain the equation determining $z_t(\vartheta)$.

The particular solution of the nonhomogeneous system

$$\frac{dz_t(\vartheta)}{dt} = Az_t(\vartheta) + F(t,\vartheta) \tag{4.13}$$

where $F(t, \vartheta)$ is a piecewise continuous function of ϑ with discontinuities of the first kind, $F(t, \vartheta) \in L$ for any $t > 0$, and is determined by Cauchy's formula

$$Z_t^* (\vartheta) = \int_0^t T (t - t_1) F (t_1, \vartheta)\, dt_1 \tag{4.14}$$

Here, $T(t)$ determines the solution $z_t(\vartheta) = T(t) Z_0(\vartheta)$ of the homogeneous system (4.13) for $F = 0$. If $z_0(\vartheta) \in L$, then $T(t) z_0(\vartheta)$ decreases exponentially:

$$T (0)\, z_0 (\vartheta) = Z_0 (\vartheta)$$

We replace the differential and operator equation (4.10) by the integral equation

$$z_t (\vartheta) = T (t)\, Z_0 (\vartheta) + \mu \int_0^t T (t - t_1)\, R_1 (y (t_1), Z_{t_1} (\vartheta)\, dt_1 \tag{4.15}$$

in which $R_1(yz)$ is known, and $y(t)$ decreases exponentially for $t \to +\infty$. Applying the method of successive approximations, we find that, for sufficiently small $|\mu|$, $Z_t(\vartheta)$ also decreases exponentially. Thus, any solution $x_t(\vartheta) = x(t + \vartheta)$ for the system (1.1) decreases exponentially for sufficiently small $|\mu|$, $\mu \neq 0$. This was to be proved.

Note. Let us assume that the system (1.1) is the system of first approximation for a system of nonlinear equations with retardation

$$\frac{dx(t)}{dt} = ax (t) + bx (t - \tau) + \mu g (t, \mu)\, x (t) + \mu h (t, \mu)\, x (t - \tau) + X (t, x (t), x (t - \tau)) \tag{4.16}$$

where $X(t, x(t), x(t - \tau))$ is a nonlinear function of the variables $x(t)$ and $x(t - \vartheta)$, whose series expansion does not contain terms of the order smaller than two, while the coefficients are uniformly bounded functions of time t. If all the characteristic indices of the system (1.1) have negative real parts, then the motion $x = 0$ of the system (4.16) is asymptotically stable for sufficiently small $|\mu|$ ($\mu \neq 0$). If at least one of the characteristic indices of the system has a positive real part, then the motion $x = 0$ of the system (4.6) is unstable [3].

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